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Expanded Uncertainty Evaluation Taking into Account the Correlation Between Estimates of Input Quantities

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Summary:
Cases of occurrence of correlation between estimates of input quantities are considered. An expression for the effective number of degrees of freedom and kurtosis of a measurand, taking into account the correlation between the estimates of the input quantities, is derived.

Keywords: expanded uncertainty, correlation, reduction method, effective degrees of freedom, kurtosis method

Introduction

When evaluating the measurement uncertainty, one has to deal with situations where estimates of input quantities are pairwise correlated. Correlation occurs in the following cases:

1) while observing both input quantities X_l and X_k entering the model

$$Y = f(X_1, X_2, \ldots X_N) \tag{1}$$

in one measurement experiment (observed correlation);

2) if there is a dependence of both input quantities X_l and X_k on the same variable Q, which appears when using the same measuring instruments, initial values or measurement methods (assumed or logical correlation):

$$X_l = \psi_l(\Theta);\ X_k = \psi_k(\Theta).$$

A measure of the correlation dependence is the correlation coefficient $r_{l,k}$, which for these two situations must be determined, respectively, by statistical (type A) and non-statistical (type B) methods [1].

When calculating the standard uncertainty $u(y)$ of the measurand Y, the correlation between the estimates is taken into account using the well-known formula [2]:

$$u(y) = \sqrt{\sum_{j=1}^{N} c_j^2 u_j^2 + 2\, r_{l,k} c_l c_k u_l u_k}, \tag{2}$$

where c_j, j=1,2,…, N is the j-th sensitivity coefficient.

Difficulties in accounting for correlation arise when evaluating expanded uncertainty U. The latter for linearized models is defined as the product of the standard uncertainty $u(y)$ by the coverage factor k:

$$U = k u_c(y). \tag{3}$$

The coverage factor k is determined in different ways with different approaches to estimating measurement uncertainty.

GUM approach

In GUM [2], the Student's coefficient $t_p(\nu_{eff})$ for the given confidence level p and the effective number of degrees of freedom ν_{eff} is taken as the coverage coefficient k for repeated measurements.

The ν_{eff} is obtained by the Welch–Satterthwaite formula [2]:

$$\nu_{eff} = \frac{u^4(y)}{\sum_{j=1}^{N} \frac{c_j^4 u_j^4}{\nu_j}}; \tag{4}$$

where ν_j is the number of degrees of freedom of the j-th input quantity.

Expression (4) does not give a correct estimate of the number of degrees of freedom in the presence of a correlation between the input quantities.

Indeed, for a function of two correlated input quantities $Y=f(X_1,X_2)$ with an equal number of degrees of freedom $\nu_1=\nu_2=\nu$ and in the absence of uncertainties of type B, the effective number of degrees of freedom will be equal to

$$\nu_{eff} = \nu \frac{(c_1^2u_1^2 + 2r_{1,2}c_1c_2u_1u_2 + c_2^2u_2^2)^2}{c_1^4u_1^4 + c_2^4u_2^4} \quad (5)$$

and when changing $-1 \le r_{1,2} \le 1$ will vary in the range from 0 to 8ν. On the other hand, to calculate the total standard uncertainty of the presence of correlation, the reduction method can be used [3]. It provides for bringing indirect measurements to direct ones by calculating the values of the measured value for each pair of correlated input quantities:

$$y_i = f(x_{1i}, x_{2i}),\ i = 1,2,...,n. \quad (6)$$

In this case, the measured value will be the arithmetic mean of the measured values obtained:

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i, \quad (7)$$

and the standard uncertainty of type A of the measured quantity is found as:

$$u_A(y) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(y_i - \bar{y})^2} \quad (8)$$

and has the number of degrees of freedom $\nu = n-1$, which should be equal to the number of degrees of freedom ν_{eff}, determined by the Welch–Satterthwaite formula (5).

This situation can be changed when taking into account that the correlated input quantities must be described by the joint PDF [4], which contributes $u_{l,k}(y)$ to the standard uncertainty of the measurand with the number of degrees of freedom $\nu = n-1$.

In this case, the expression for the combined standard uncertainty (2) can be rewritten as follows:

$$u(y) = \sqrt{\sum_{\substack{j=1 \\ j \ne k \ne l}}^{N} c_j^2u_j^2 - u_{l,k}^2(y)}, \quad (9)$$

from which

$$u_{l,k}(y) = \sqrt{c_l^2u_l^2 + 2r_{l,k}c_lc_ju_lu_k + c_k^2u_k^2} \quad (10)$$

In this case, the Welch–Satterthwaite formula in the presence of correlated input quantities will have the form:

$$\nu_{eff} = \frac{u^4(y)}{\sum_{\substack{j=1 \\ j \ne k \ne l}}^{N} \frac{c_j^4u_j^4}{\nu_j} + \frac{u_{l,k}^4}{n-1}}, \quad (11)$$

So, for a function of two correlated input quantities $Y=f(X_1,X_2)$ with an equal number of degrees of freedom $\nu_1=\nu_2=\nu$, the effective number of degrees of freedom will be equal to ν, which coincides with the number of degrees of freedom for expression (8).

Bayesian approach. Kurtosis method

Expanded uncertainty:

$$U = k(\eta) \cdot u(y), \quad (12)$$

where the coverage factor $k(\eta)$ depends on the kurtosis η of the measurand, determined by the formula:

$$\eta = \frac{\sum_{j=1}^{N} c_j^4u_j^4\eta_j}{u^4(y)}, \quad (13)$$

where η_j is kurtosis of the j-th input quantity.

This expression also does not work in the case of correlated input quantities, however, it can be transformed for this case by analogy with expression (11):

$$\eta = \frac{\sum_{\substack{j=1 \\ j \ne k, j \ne l}}^{N} c_j^4u_j^4\eta_j + \eta_{l,k}u_{l,k}^4(y)}{u^4(y)}. \quad (14)$$

The coverage factor for a confidence level of 0.95 is calculated by the formula [5]:

$$k = \begin{cases} 0.1085\eta^3 + 0.1\eta + 1.96, & \text{when } \eta < 0; \\ t_{0.95;(6/\eta+4)} \cdot \sqrt{\frac{3+\eta}{3+2\eta}}, & \text{when } \eta \ge 0. \end{cases} \quad (15)$$

Examples of evaluation the expanded uncertainty of various measurements are considered, taking into account the correlation between estimates of input quantities.

References

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