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The Optimal Axial Strain Distribution in a Piezoelectric Vibrating Energy Harvester (PVEH)

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Summary:

This study considers the optimal strain distribution in a piezoelectric vibrating energy harvester (PVEH), that maximizes the harvested energy. In many previous studies it was tacitly assumed that a uniform distribution of axial strain in the piezoelectric layer, ensures that the harvested energy is maximal. Though this assumption is intuitive, it was not supported by analysis. In this work we derive and present a formal analytic proof confirming that, for a given amount of energy in the vibrating structure, a uniform distribution of strain in the piezoelectric layer ensures that the harvested energy is maximal.

Keywords: Piezoelectric vibrating energy harvester (PVEH), piezoelectric unimorph, energy harvesting, strain distribution, variational analysis.

Background, Motivation and Objective

Piezoelectric vibrating energy harvester (PVEH) devices have been studied for over two decades [1], [2], [3]. The most prevalent PVEHs are constructed from a piezoelectric unimorph, in which an elastic cantilever is coated by a thin layer of piezoelectric material. This piezoelectric layer is sandwiched between a top and bottom electrode. When the piezoelectric unimorph is subjected to base excitations at its natural frequency, the amplitude of vibrations increases.

In previous studies it has been identified that an optimal planform of the unimorph cantilever may result in a uniform amplitude of axial strain in the piezoelectric layer. It was further assumed that such a uniform amplitude of strain maximizes the harvested energy. This seems to be intuitively sensible, but no formal proof of this has been provided.

The aim of the present study is to provide a formal proof, and show that for a given amount of vibration energy in a PVEH, a *uniform* distribution of axial strain ensures that the harvested energy is maximal.

Modelling – Analytic derivation

Figure 1 presents a schematic illustration of a piezoelectric cantilever unimorph, that is constructed from an elastic substrate of thickness h, coated with a thin piezoelectric layer of thickness h_{pe} (h_{pe}<<h). The beam length is L and the width $b(x)$ may be non-uniform.

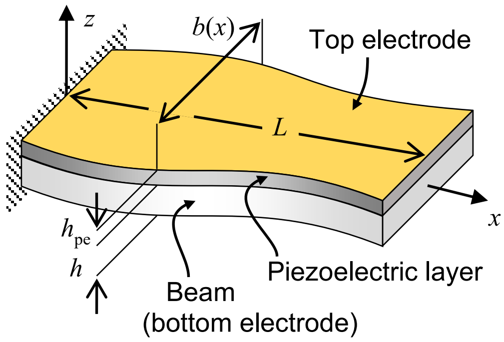

Fig. 1. Schematic description of a PVEH unimorph cantilever.

the electric energy U_E that may be harvested in one half of a motion cycle, is proportional to the product of the charge amplitude Q_{Short} that is transferred between the electrodes – if they are shorted (i.e. connected), and the amplitude of voltage difference V_{Open} between the electrodes – if they are open (i.e. disconnected) [4], [5]:

$$U_E = \frac{1}{2} Q_{\text{Short}} V_{\text{Open}} \tag{1}$$

In the piezoelectric layer, poling points in the z direction, such that material directions 1, 2 and 3, are parallel to the spatial axes x, y and z, respectively. According to the Euler-Bernoulli beam theory, and considering that the piezoelectric layer is thin, we may deduce that within the piezoelectric layer several terms vanish.

Within this layer, in terms of the Voight notation, the shear strains and stresses are zero (i.e. $S_4=S_5=S_6=0$ and $T_4=T_5=T_6=0$), and the transverse stress T_3 is zero because the top surface of the unimorph is stress-free. Furthermore, due to geometrical considerations, the components of the electric field and electric displacement (i.e. flux) along the x and y axes vanish (i.e. $E_1=E_2=0$ and $D_1=D_2=0$). Finally, for cylindrical bending we may assume that $S_2=0$, and that S_1 is not a function of y. Because the piezoelectric layer is thin we may also consider S_1 within the piezoelectric layer to be independent of z (this is known as the small-piezoelectricity assumption [6], [7]), so that overall $S_1(x)$ is uniform in each cross-section, though it may vary along the beam.

Therefore, we get two coupled equations:

$$T_3(x) = 0 = C_{13}^E S_1(x) + C_{33}^E S_3(x) - e_{33}E_3 \quad (2)$$

$$D_3(x) = e_{31}S_1(x) + e_{33}S_3(x) + \varepsilon_{33}^S E_3 \quad (3)$$

By imposing electrostatic boundary conditions, we derive Q_{Short} and V_{Open}

$$Q_{\text{Short}} = \left(e_{31} - \frac{e_{33}C_{13}^E}{C_{33}^E}\right)\int_{x=0}^{L} S_1(x)b(x)dx \quad (4)$$

$$V_{\text{Open}} = -h_{pe}\left[\frac{C_{13}^E}{e_{33}} - \frac{C_{33}^E}{e_{33}}\frac{\left(e_{31}+\varepsilon_{33}^S\frac{C_{13}^E}{e_{33}}\right)}{\left(e_{33}+\varepsilon_{33}^S\frac{C_{33}^E}{e_{33}}\right)}\right]\frac{\int_{x=0}^{L} S_1(x)b(x)dx}{\int_{x=0}^{L} b(x)dx} \quad (5)$$

Substituting these expressions from (4) and (5) into (1), yields

$$U_E = \alpha\left(\int_{x=0}^{L} S_1(x)b(x)dx\right)^2 \Big/ \int_{x=0}^{L} b(x)dx \quad (6)$$

where α is a scalar that depends on material parameters and the thickness of the piezoelectric layer:

$$\alpha = -\frac{h_{pe}}{2}\left(e_{31} - \frac{e_{33}C_{13}^E}{C_{33}^E}\right)\left[\frac{C_{13}^E}{e_{33}} - \frac{C_{33}^E}{e_{33}}\frac{\left(e_{31}+\varepsilon_{33}^S\frac{C_{13}^E}{e_{33}}\right)}{\left(e_{33}+\varepsilon_{33}^S\frac{C_{33}^E}{e_{33}}\right)}\right] \quad (7)$$

The mechanical elastic energy stored in the beam due to bending is given by

$$U_M = \beta\int_{x=0}^{L} S_1^{\,2}(x)b(x)dx \quad (8)$$

where $\beta = E_Y h/6$ and E_Y is Young modulus.

We wish to find the axial strain distribution function $S_1(x)$, which will maximize the electrostatic energy U_E for an arbitrary distribution of width $b(x)$. However, we aim to find the maximal electrostatic energy for all possible distributions on $S_1(x)$, which generate the same specific mechanical elastic energy U_M^*.

This can be done by considering a constrained Lagrangian, with a Lagrange multiplier λ and a functional J_λ

$$J_\lambda[S_1(x),\lambda] = U_E - \lambda\cdot\left(U_M - U_M^*\right) \quad (9)$$

Applying the first variation for the functional with respect to $S_1(x)$ and λ, yields

$$\delta J_\lambda = 2\int_{x=0}^{L}\left\{\alpha\frac{\int_{x'=0}^{L} S_1(x')b(x')dx'}{\int_{x'=0}^{L} b(x')dx'} - \lambda\beta\cdot S_1(x)\right\}b(x)\cdot\delta S_1(x)dx - \delta\lambda\cdot\left\{\beta\int_{x=0}^{L} S_1^{\,2}(x)b(x)dx - U_M^*\right\} \quad (10)$$

Our goal is to identify the function form of $S_1(x)$ at which the electrostatic energy reaches its maximum. Therefore, according to (10), the variation must be equal to zero for any $\delta S_1(x)$ and any $\delta\lambda$. This requires that each one of the expressions that appear in curly brackets vanish, given that $b(x)$ cannot be zero. It follows that:

$$S_1(x) = \sqrt{U_M^* \Big/ \beta\int_{x=0}^{L} b(x)dx} \quad (11)$$

This confirms that the strain within the piezoelectric layer must be uniform.

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