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Methods for Computing Quasi-Guided Waves in Plate Structures Coupled to Unbounded Media

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Abstract: Using semi-analytical methods for modeling waves in layered structures, the thickness direction is discretized straightforwardly by finite elements. However, the coupling to adjacent acoustic or elastic half-spaces leads to a nonlinear eigenvalue problem (NLEP) for the wavenumbers and mode shapes. We demonstrate how this NLEP can be solved in two very different ways, either by recasting it as a multiparameter eigenvalue problem or by the concept of exponential residual relaxation in conjunction with standard Runge-Kutta solvers.

Keywords: guided waves, semi-analytical methods, leaky waves, multiparameter eigenvalue problems, Zhang Neural Networks

Problem statement

We address the problem of elastic waves propagating along a plate of constant thickness h with a (generally complex) wavenumber k, see Fig. 1. If the plate surfaces are traction-free or fixed, this setup corresponds to the classic Lamb wave problem [1], and various methods exist for the computation of wave modes and dispersion curves; see, e.g., [2, 3] and the references therein. In particular, it is common to discretize the plate thickness using finite elements, yielding a quadratic eigenvalue problem for the wavenumbers at a given frequency ω. If, on the other hand, the plate is in contact with fluid or solid half-spaces at either or both of its surfaces, the situation becomes significantly more complicated, as the interaction with the half-spaces results in additional nonlinear terms in the eigenvalue problem, involving the free-field longitudinal and transversal wavenumbers κ, γ in the half-spaces [4, 5, 6]. Specifically, the nonlinear eigenvalue problem is of the form

$$\left(\mu\,\mathbf{M} - \mathbf{E}_2 + \mathrm{i}k\,\mathbf{E}_1 - k^2\,\mathbf{E}_0 + \mathbf{R}(k,\mu)\right)\boldsymbol{\phi} = \mathbf{0} \quad (1)$$

which we abbreviate as

$$\mathbf{L}(k,\mu)\,\boldsymbol{\phi} = \mathbf{0} \quad (2)$$

with $\mu = \omega^2$. Here, $\mathbf{M}, \mathbf{E}_0, \mathbf{E}_1, \mathbf{E}_2$ are finite element matrices, and $\boldsymbol{\phi}$ denotes the eigenvector, representing the amplitudes of displacements in the plate as well as the pressure or displacements in the unbounded domains. Furthermore, $\mathbf{R}(k,\mu)$ denotes the nonlinear terms stemming from the coupling to the half-spaces:

$$\mathbf{R}(k,\mu) = \sum_{j=1}^{6} b_j\,\xi_j\,\mathbf{R}_j \quad (3)$$

with

$$b_j = \begin{cases} \mathrm{i} & j \in \{1,2\} \quad \text{fluid half-space,} \\ k & j \in \{3,4,5,6\} \quad \text{solid half-space.} \end{cases} \quad (4)$$

We denote by ξ the vertical wavenumbers of partial waves in the unbounded domains:

$$\xi_j(k,\mu) = \pm\sqrt{\frac{\mu}{c_j^2} - k^2} \quad (5)$$

with the corresponding wave speed c_j. The notation in Eq. (3) includes all six possible partial waves, while for any half-space not present in the current model, we set the corresponding $\mathbf{R}_j$ matrices to $\mathbf{0}$.

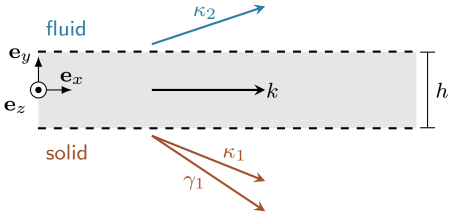


Fig. 1: Schematic of a homogeneous plate of thickness h between a fluid and a solid half-space.

Solving the multiparameter eigenvalue problem

Equation (1) can be posed as a linear *multiparameter* eigenvalue problem in the parameters k, k^2, ξ_j. For brevity, we show the simplest case involving only one partial wave. This scenario may represent a plate coupled to a fluid either on one surface only or the same fluid on both surfaces. In this case, solutions to Eq. (1) satisfy the following system:

$$\begin{gathered}(\mu\,\mathbf{M} - \mathbf{E}_2 + \mathrm{i}k\,\mathbf{E}_1 + \xi_0\,\mathbf{E}_0 + \mathrm{i}\xi_1\mathbf{R}_1)\,\phi = \mathbf{0},\\ \left(\begin{bmatrix}0 & -\kappa_1^2\\ 1 & 0\end{bmatrix} + \mathrm{i}\xi_1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \xi_0\begin{bmatrix}0 & -1\\ 0 & 0\end{bmatrix}\right)\mathbf{x}_1 = \mathbf{0},\\ \left(\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + \mathrm{i}k\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \xi_0\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right)\mathbf{x}_2 = \mathbf{0},\end{gathered}$$

where

$$\xi_1^2 := \kappa_1^2 - k^2, \quad \xi_0 = -k^2 \tag{6}$$

and $\mathbf{x}_1 \neq 0$, $\mathbf{x}_2 \neq 0$. Note that the determinants of the matrices in the second and third equations equal $\kappa_1^2 - k^2 - \xi_1^2$ and $\xi_0 + k^2$, respectively; hence, these equations incorporate the relations (6). If the half-space consists of a solid elastic material, the system is slightly different. Considering again only one partial wave in the unbounded domain, we obtain

$$\begin{gathered}(\mu\,\mathbf{M} - \mathbf{E}_2 + \mathrm{i}k\,\mathbf{E}_1 + \xi_0\,\mathbf{E}_0 + k\xi_3\mathbf{R}_1)\,\phi = \mathbf{0},\\ \left(\begin{bmatrix}0 & -\kappa_1^2\\ 0 & 0\end{bmatrix} + k\xi_3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \xi_0\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right)\mathbf{x}_1 = \mathbf{0},\\ \left(\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + \mathrm{i}k\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \xi_0\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right)\mathbf{x}_2 = \mathbf{0}.\end{gathered}$$

Introducing several partial waves or both a solid and a fluid half-space coupled to the same plate can be achieved straightforwardly by extending the system of equations accordingly. Details are found in [5]. Nowadays, established algorithms exist for the solution of multiparameter eigenvalue problems of this type [7, 8]. We make use of the approach implemented in the Matlab toolbox MultiParEig [9]. It requires the solution of a linear generalized eigenvalue problem involving the so-called operator determinants, which are constructed from Kronecker products of the matrices defining the multiparameter eigenvalue problem. While this approach allows the accurate computation of all solutions at any given frequency, the computational costs can be significant, as the operator determinants are square matrices of size $2^{n_\xi+1}n$, where n denotes the size of the finite element matrices, and n_ξ is the total number of partial waves in the half-spaces. Hence, in the most challenging case of a plate coupled to two different elastic half-spaces, the operator determinants are of size $32n \times 32n$.

Solution by exponential residual relaxation

The concept of exponential residual relaxation, also known as Zhang Neural Networks [10, 11, 12], is the following. Say we want to minimize some parameterized objective function, i.e., we are interested in solving

$$\mathbf{f}(\mathbf{y}(\mu), \mu) = 0 \tag{7}$$

with a given parameter μ, defined on some interval $\mathcal{I}$. The idea is to postulate an exponentially decreasing residual of the above equation, such that

$$\mathbf{f}'(\mathbf{y}(\mu), \mu) = -\chi\,\mathbf{f}(\mathbf{y}(\mu), \mu), \quad \mu \in \mathcal{I} \qquad \text{(8)a}$$

$$\mathbf{y}(\mu_0) = \mathbf{y}_0, \qquad \text{(8)b}$$

where $\mathbf{f}'$ denotes the derivative with respect to μ, and χ is an algorithmic constant, defining the decay rate. In this form, Eqs. (8) pose an initial value problem that can be solved using standard algorithms, typically of the Runge-Kutta type. To apply this idea to our nonlinear eigenvalue problem, we define the objective function

$$\mathbf{f}(\phi, k, \mu) = \begin{bmatrix}\mathbf{L}(k, \mu)\,\phi\\ \phi^{\mathrm{H}}\phi - 1\end{bmatrix}, \tag{9}$$

where the second row introduces a normalization required to obtain unique eigenvectors. The total derivative with respect to the parameter μ is given as

$$\mathbf{f}'(\phi, k, \mu) = \begin{bmatrix}\mathbf{L}(k, \mu)\phi' + \mathbf{L}'(k, \mu)\phi\\ 2\phi^{\mathrm{H}}\,\phi'\end{bmatrix}. \tag{10}$$

Substituting $\mathbf{f}$ and $\mathbf{f}'$ into Eq. (8)a yields

$$\begin{bmatrix}\mathbf{L} & \mathbf{L}_{,k}\,\phi\\ 2\phi^{\mathrm{H}} & 0\end{bmatrix}\begin{bmatrix}\phi'\\ k'\end{bmatrix} = -\begin{bmatrix}\chi_1\,\mathbf{L}\,\phi + \mathbf{L}_{,\mu}\,\phi\\ \chi_2\,(\phi^{\mathrm{H}}\phi - 1)\end{bmatrix}, \tag{11}$$

which can be solved numerically starting from some known solution ϕ_0, k_0. The decay parameters are chosen as

$$\chi_1 = 100\frac{h^2}{c_t^2}, \quad \chi_2 = 10\frac{h^2}{c_t^2} \tag{12}$$

with h and c_t denoting the plate's thickness and transversal wave speed, respectively. This approach requires reasonably accurate initial values at some μ_0, which can be computed either by the method described in the previous section or by suitable approximations [6], such as those obtained by replacing the exact boundary conditions by simple dashpots [13]. To solve Eq. (11), we employ a Runge-Kutta-based solver optimized for stiff differential equations with a variable step size implemented in Matlab's function *ode15s*.

Numerical example

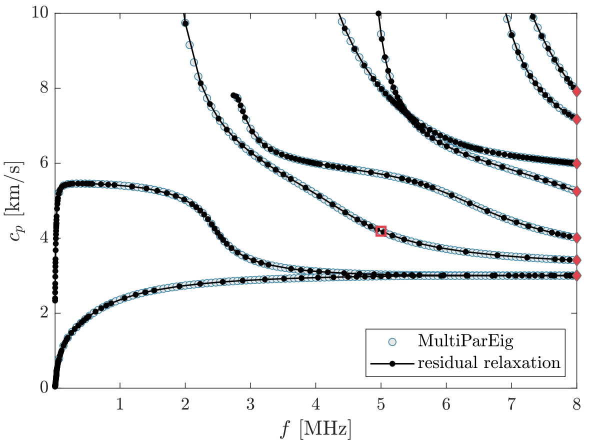
To demonstrate the capabilities of both approaches and compare their results, we present a challenging yet somewhat academic example, namely a titanium plate between an acoustic and an elastic half-space, consisting of water and Teflon, respectively. The material parameters are listed in Tab. 1. Figure 2 shows the phase velocity and attenuation of lowly attenuated modes (less than 4 dB/mm) up to a frequency of 8 MHz, computed using both proposed algorithms. For conciseness, we only show those solutions for which all wave vectors in the unbounded domains point away from the plate. In addition, there are solutions with one or more wave vectors pointing towards one of the plate surfaces. In practical applications, one is usually more interested in the direction of the power flux, which can be used as a criterion for distinguishing modes, see [5]. Applying the solution procedure based on multiparameter eigenvalue problems, we compute solutions at 120 predefined frequency steps, resulting in a total computational time of about 30 s on a current laptop computer. Employing the concept of exponential residual relaxation, we use the solution obtained by the first approach at 8 MHz and trace each of the shown modes towards $\mu = 0$. This computation takes about 1 s in total and requires an average of 74 steps per mode as chosen automatically by the solver when setting the solver's relative tolerance to 0.01. The results of both approaches are in excellent agreement. In addition, we show the wave field of an arbitrarily selected mode, which is obtained directly by computing the eigenvector ϕ, as it includes the amplitudes of displacements and pressure both inside the plate and in the unbounded domains. Specifically, we selected the third propagating mode (roughly corresponding to the A_1 mode in a free plate). Note that, while the results directly provide amplitudes along the plate's thickness, the wave field anywhere inside the plate and the unbounded domains can be computed in a postprocessing step. A detailed discussion of the accuracy and performance, as well as further examples, can be found in [5, 6]. The method based on multiparameter eigenvalue problems is also implemented in the free Matlab toolbox SAMWISE [14].

Tab. 1: Material parameters used in the example.

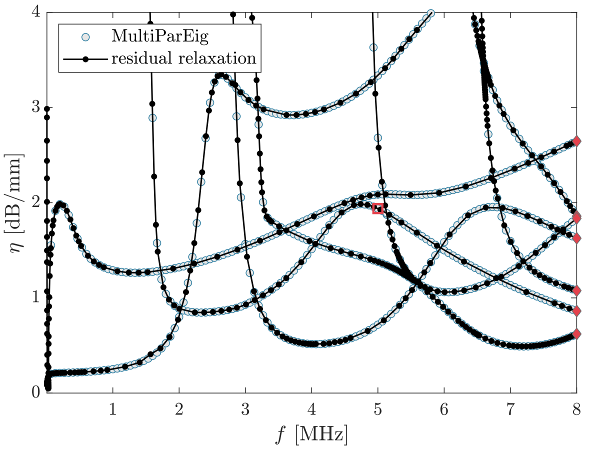
	density ρ	wave speeds c_ℓ, c_t	
Teflon	2.20 g/cm^3	1.35 km/s	0.55 km/s
titanium	4.46 g/cm^3	6.06 km/s	3.23 km/s
water	1.00 g/cm^3	1.48 km/s	

Conclusion

The proposed approaches help overcome a previous obstacle in applying semi-analytical methods to the case of plate structures coupled to unbounded domains. We provide two highly robust methods for solving the nonlinear eigenvalue problems that arise in this context. These techniques allow the computation of mode shapes, wavenumbers, and modal attenuation for arbitrarily layered plate structures coupled to solid and/or acoustic fluid media.



(a) phase velocity



(b) attenuation

Fig. 2: Dispersion curves of a 1-mm-thick titanium plate between a solid and a fluid half space. Results are obtained using the framework of multiparameter eigenvalue problems as well as the proposed approach based on exponential residual relaxation. The symbol □ marks the solution selected for plotting the wave field in Fig. 3.

References

[1] H. Lamb. "On waves in an elastic plate". In: *Proceedings of the Royal Society of London* 93.648 (1917), pp. 114–128. DOI: 10.1098/rspa.1917.0008.

[2] M. J. S. Lowe. "Matrix techniques for modeling ultrasonic waves in multilayered media". In: *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control* 42.4 (1995), pp. 525–542. DOI: 10.1109/58.393096.

[3] H. Gravenkamp, C. Song, and J. Prager. "A numerical approach for the computation of dispersion relations for plate structures using the scaled boundary finite element method". In: *Journal of Sound and Vibration* 331.11 (2012), pp. 2543–2557. DOI: 10.1016/j.jsv.2012.01.029.

[4] H. Gravenkamp, C. Birk, and C. Song. "Numerical modeling of elastic waveguides coupled to infinite fluid media using exact boundary conditions". In: *Computers & Structures* 141 (2014), pp. 36–45. DOI: 10.1016/j.compstruc.2014.05.010.

[5] H. Gravenkamp et al. "Computation of leaky waves in layered structures coupled to unbounded media by exploiting multiparameter eigenvalue problems". In: *Journal of Sound and Vibration* 596 (2025), p. 118716. DOI: 10.1016/j.jsv.2024.118716.

[6] H. Gravenkamp, B. Plestenjak, and D. A. Kiefer. "Computing leaky waves in semi-analytical waveguide models by exponential residual relaxation". In: *ArXiv e-prints* (2025). DOI: 10.48550/arXiv.2505.04735.

[7] B. Plestenjak, C. I. Gheorghiu, and M. E. Hochstenbach. "Spectral collocation for multiparameter eigenvalue problems arising from separable boundary value problems". In: *Journal of Computational Physics* 298 (2015), pp. 585–601. DOI: 10.1016/j.jcp.2015.06.015.

[8] E. Ringh and E. Jarlebring. "Nonlinearizing two-parameter eigenvalue problems". In: *SIAM Journal on Matrix Analysis and Applications* 42.2 (2021), pp. 775–799. DOI: 10.1137/19M1274316.

[9] B. Plestenjak. *MultiParEig*. 2023. URL: www.mathworks.com/matlabcentral/fileexchange/47844-multipareig.

[10] Y. Zhang and J. Wang. "Recurrent neural networks for nonlinear output regulation". In: *Automatica* 37.8 (2001), pp. 1161–1173. DOI: 10.1016/S0005-1098(01)00092-9.

[11] F. Uhlig and Y. Zhang. "Time-varying matrix eigenanalyses via Zhang Neural Networks and look-ahead finite difference equations". In: *Linear Algebra and Its Applications* 580 (2019), pp. 417–435. DOI: 10.1016/j.laa.2019.06.028.

[12] F. Uhlig. "Zhang neural networks: An introduction to predictive computations for discretized time-varying matrix problems". In: *Numerische Mathematik* 156 (2024), pp. 691–739. DOI: 10.1007/s00211-023-01393-5.

[13] H. Gravenkamp, C. Birk, and C. Song. "Computation of dispersion curves for embedded waveguides using a dashpot boundary condition". In: *The Journal of the Acoustical Society of America* 135.3 (2014), pp. 1127–1138. DOI: 10.1121/1.4864303.

[14] H. Gravenkamp. *SAMWISE: Semi-Analytical Modeling of Waves in Structural Elements*. 2024. DOI: 10.5281/zenodo.13830671. URL: https://github.com/haukegravenkamp/SAMWISE.

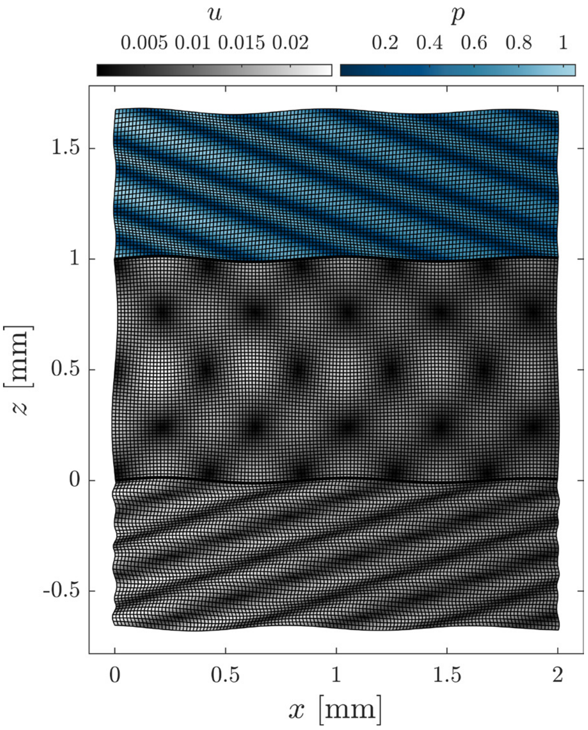


Fig. 3: Wave field of the third propagating mode at a frequency of 5 MHz.