DOI 10.5162/Ultrasonic2025/A8-a2

Quaternion Differential Equation for the $(2+1)D$ and $(3+1)D$ NDT

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Abstract: In Non-Destructive Testing (NDT), Ultrasonic (US) wave scattering positions are detected by interference of reflected original wave and reflected time-reversed (TR) wave. TR-US waves are produced by using the memristor. Transducers and detectors are located in a 2D plane and hysteresis effects are measured. We analyze the spectrum data of Wire Arc Additive Manufacturing (WAAM) sample obtained by Quaternion Excitation Symmetry Analysis Method (QESAM) using the conformally invariant quantum mechanical variables of de Alfaro-Fubini-Furlan. Depending on the relative position of the transducer and receivers, we observed conformality of spectra on different receivers. We review phonetic wave analysis of memristors, and show our lattice simulation data, which could contain chaotic behaviors. The (2+1)D analysis is extended to (3+1)D, by replacing quaternions to biquaternions and split-quaternions and discuss their lattice simulation

Keywords: Nonlinear Elastic Wave Spectroscopy, Biquaternion, Split-Quaternion, Conformal Symmetry, Encryption/Decryption

Introduction

In the Time Reversal based Nonlinear Elastic Wave Spectroscopy(TR-NEWS) [1] we use the non-commutativity of Quaternions to take into account nonclassical nonlinearity (hysteresis properties) of US wave propagation on a 2D plane. In the relativistically invariant propagation of waves in (1+1)D, Tomonaga[2] proposed defining time as a function of 3D space. In (3+1)D Quantum Chromo Dynamics, DeGrand et al.[3] simulated propagation of gluons in Dirac fermions sitting on 3D lattice sites using the Fixed Point action. We replaced gluons to phonons and Dirac fermions to Weyl fermion and simulated (2+1)D wave propagation[4].

In analysis of stochastic processes expressed by correlation/convolution processes, it is necessary to optimize the US wave propagation in materials. The stochastic processes follow the Langevin equation [5], and Lattice simulations can be applied[6].

The optimization of paths can be achieved by the minimal action, and we use Machine Learning tools for an extended Excitation Symmetry Analysis Method (ESAM) [7] signal processing.

(2+1)D US wave propagation

We perform WAAM sample data analysis by TR-NEWS. US waves produced by a transducer T_X are scattered in materials, and signals $s_1(t) = N_1 x(t), s_2(t) = N_2 x(t)^2$, $s_3(t) = N_3 x(t)^3$ are received by 12 receivers $R_1, \cdots, R_{12}$ The received signal R_6 and R_{12} are shown in Fig.2 de Alfaro, Fubini and Furlan[8] considered (1+1)D Lagrangian $L = \frac{1}{2}(\dot{Q}^2 + \frac{g}{Q^2})$ and Hamiltonian $H = \dot{Q}\frac{\partial L}{\partial \dot{Q}} L$, dilatation D and conformal operator K obeys the algebra

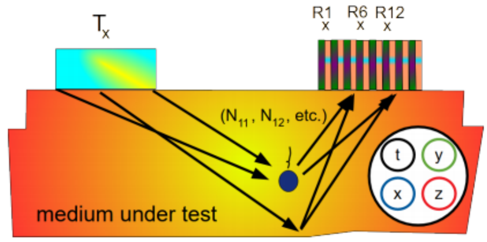

Fig. 1: The experimental setup of TR-NEWS using WAAM sample.

$$[H, D] = \sqrt{-1}H, [K, D] = -\sqrt{-1}K, [H, K] = 2\sqrt{-1}D \quad (1)$$

and constructed operators of $O(2,1)$ group by introducing R and S defined as

$$R = \frac{1}{2}(\frac{1}{a}K + aH), S = \frac{1}{2}(\frac{1}{a}K - aH) \quad (2)$$

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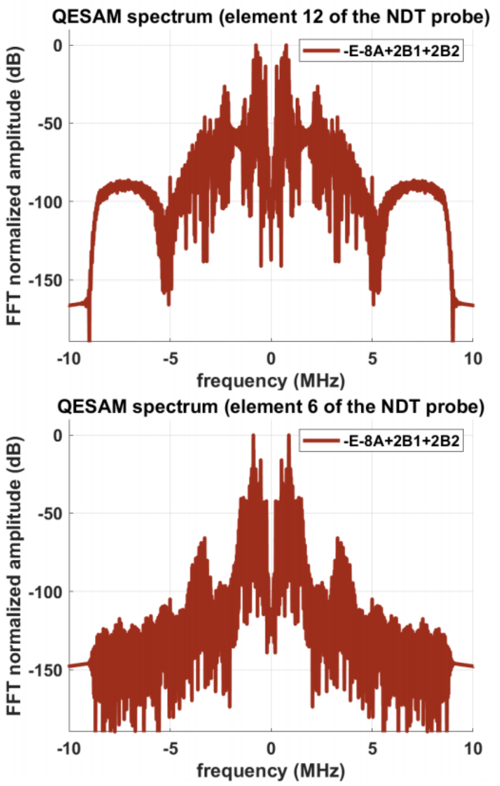

Fig. 2: The spectra of QESAM received at R12(top) and at R6(bottom).

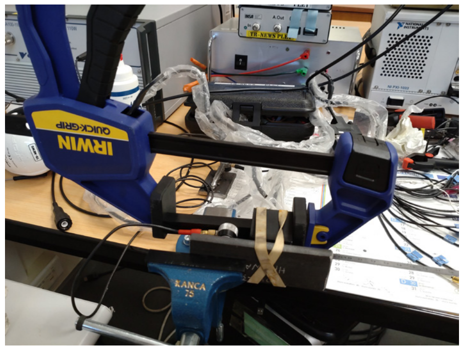

Fig. 3: The experimental setup.

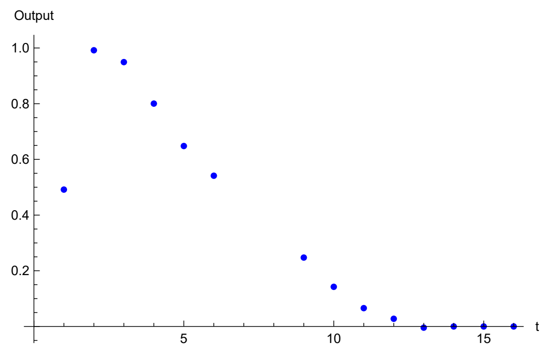

Fig. 4: Output calculated by the weight function at the 4000th cycle and the fixed point action at $t = 2, \cdots, 16$. The outputs in the range $t = 2, \cdots, 6$ are shifted upward by 0.5.

where a is a constant length, and conformal relation

$$[D, R] = \sqrt{-1}S, [S, R] = -\sqrt{-1}D, [S, D] = -\sqrt{-1}R \quad (3)$$

Godberashvili[9] defined a split quaternion

$$\begin{aligned} q &= q_0 + q_1 i + q_2 j + q_3 k \\ &= q_0 + q_1 i + (q_2 + q_3 i)j \\ q^* &= q_0 - q_1 i - q_2 j - q_3 ij \end{aligned} \quad (4)$$

In (2+1)D, $k = ij$ can be taken as the time axis, which induces the conformal $O(2, 1)$ symmetry.

(3+1)D US wave propagation

The Godberashvilli's formula can be extended to

$$\begin{aligned} q &= q_0 + q_2 j + (q_3 + q_1 j)k \\ q^* &= q_0 - q_2 j - q_3 k - q_1 jk \\ q &= q_0 + q_3 k + (q_1 + q_2 k)i \\ q^* &= q_0 - q_3 k - q_1 i - q_2 ki \end{aligned} \quad (5)$$

Garling[10] defined biquaternions from quaternions e_1, e_2, e_3, e_4 as e_2e_3, e_3e_1, e_1e_2 as spacelike unit vectors and e_1e_4, e_2e_4, e_3e_4 as spacelke time.

In the (3+1)D lattice simulation, there are time shifts $\pm e_i e_4 (i = 1, 2, 3)$ on each paths that start from origin and return to the origin in 16 steps. There are 7 paths and their weight functions are fixed by an extension of Echo State Network(ESN). Extension means, the matrices that connect reservoir and output W_{ro} is obtained from the Fixed Point action and not fixed as in standard ESN,

The output of ESN are stable after 3000th cycle till 4000th cycle. The output has the Möbius band structure. When the outputs of $0 < t < 8$ are shifted upward by 0.5, the whole outputs become smooth [11].

Chaos in (3+1)D and its application

In (3+1)D, there are chaotic paths, which can be used in encrypting and decrypting[12]. We solved the differential equation of Chen's system using Mathematica and from $x(t) = (x_1(t), x_2(t), x_3(t))$ obtained masked signal $s_M(t)$ and encrypted signal $e(s_M(t)) = (f_1(f_1(f_1(f_1(s_M(t), k(t)), k(t)), k(t)), k(t)) = c(t)$

where

$$f_1(x,k) = \begin{cases} (x+k)+2h, & -2h \le (x+k) \le -h \\ (x+k), & (x+k) < h \\ (x+k)-2h, & h \le (x+k) \le 2h. \end{cases} \quad (6)$$

The decrypted signal is $s_R(t) = d(c(t)) = f_1(f_1,(f_1(f_1(c(t),-\hat{k}(t)),-\hat{k}(t)),-\hat{k}(t)),-\hat{k}(t))$. $s_M(t)$ and $s_R(t)$ are sensitive to the initial conndition, and in a certain range, the difference of $s_M(t)$ and $s_R(t)$ are negligible.

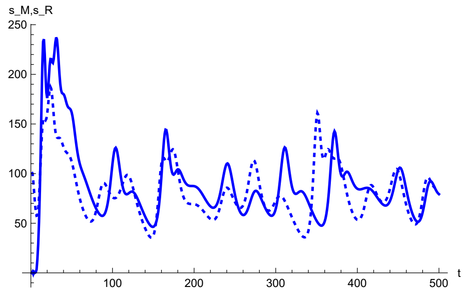

Fig. 5: The 4th step $s_M(t) = s_R(t)$ of Chen system. Solid line is calculated from $x(t)$, dashed line is calculated from $\hat{x}(t)$. $420 \le t \le 500$ can be used for encrypting/decrypting.

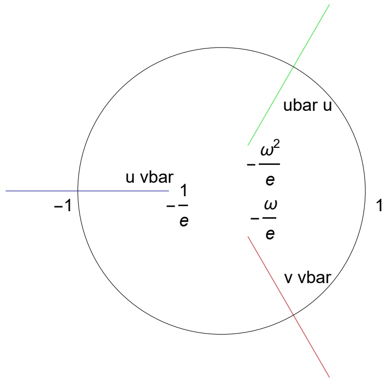

Fig. 6: The complex plane with three branch lines $z = u\bar{v}$, $z = v\bar{v}$ and $z = \bar{u}u$.

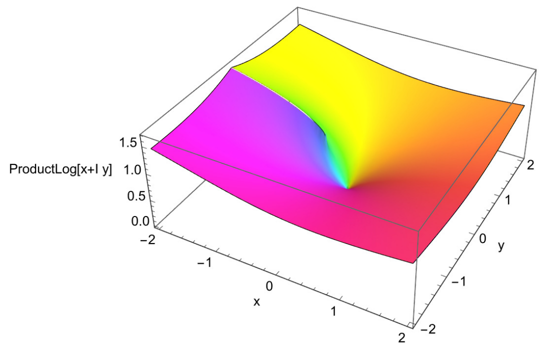

Fig. 7: The 3D plot of the ProductLog[z] function.

Dirac's bi-quaternion and Triality

Dirac applied Lorentz transformation to quaternions[13]. He expressed bi-quaternion as a ratio $q = uv^{-1}$, and for any quaternion λ, $q = u\lambda\lambda^{-1}v^{-1} = u\lambda(v\lambda)^{-1}$. Let $Q_1 = u\bar{v}$ where $\bar{v}$ is the conjugate quaternion, '$Q_2 = u\bar{u}$, $Q_3 = v\bar{v}$, and 4D vectors X_1, X_2, X_3 and X_4 are related by $Q_1 = X_0+X_1 i+X_2 j+X_3 k$, $Q_2 = X_4-X_5$, $Q_3 = X_4+X_5$. The relatiion $Q_1\bar{Q}_1 = u\bar{v}v\bar{u} = uQ_3\bar{u} = Q_2 Q_3$ leads to a constraint $X_0^2+X_1^2+X_2^2+X_3^2 = X_4^2-X_5^2$

Elgindy[14] tranformed the parameter $X_4 = t$ to $\tau = t(1-y^{1/\alpha})$ or $t-\tau = ty^{1/\alpha}$ and proposed use of fractional-order shifted Gegenbauer polynomial $\hat{G}_\alpha^\lambda(\tau)$ instead of Gegenbauer polynomial $G_\alpha^\lambda(x)$ with weight function $w^\lambda(x) = (1-x^2)^{\lambda-1/2}$.

The branch of Gegenbauer Polynomial is analogous to that of Productlog[z] function of Mathematica, which is the main solution of $z = we^w$, which on a complex plane posess a cut from $-\infty$ to $-1/e$ as shown in Fig.7. The split quaternion suggests the interchange of u to v, $\bar{v}$ to $\bar{u}$ changes the branch line of the ProductLog[z] function. Existence of three symmetric states is not equal to triality of octonions, we need to optimize the path on the multi-complex plane.

Hysteresis from Stochastic Differential Equation

In Preisach-Mayergoyz model[4], for input u, we define $x_n = u + X_n$, including the noise X_n., The output f_t is given by a solution of Itō's stochastic differential equation[15]

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad (7)$$

where $b(X_t) = \delta\dot{X}_t$ and $\sigma(X_t)$ is a nondegenerate square matrix.

The output is

$$f_t = \int\int_{\alpha\ge\beta} \mu(\alpha,\beta)E\{\hat{\gamma}_{\alpha\beta}x_t\}d\alpha d\beta. \quad (8)$$

The function $\mu(\alpha,\beta)$ is approximately Gaussian function as Fig.8. The expectation value $E\{\hat{\gamma}_{\alpha\beta}x_n\} = P\{\hat{\gamma}_{\alpha\beta}x_n = +1\} - P\{\hat{\gamma}_{\alpha\beta}x_n = -1\}$. The probability $P\{\hat{\gamma}_{\alpha\beta}x_n = +1\} + P\{\hat{\gamma}_{\alpha\beta}x_n = -1\} = 1$, and its hysteresis effects can be seen for a fixed α change of f as u increases, as shown in Fig.9..

Discussion and Outlook

We showed that TR-NEWS method using quaternion is successful in (2+1)D NDT, and conformal prop-

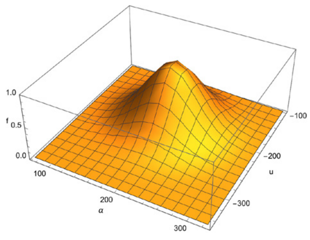

Fig. 8: The $\mu(\alpha, \beta)$ of the Preisach Model

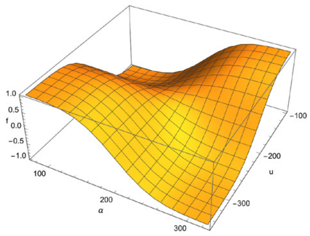

Fig. 9: The function $f(\alpha, u)$ derived from $\mu(\alpha, \beta)$

erty was observed. In (3+1)D NDT, biquaternion or split quaternion bases are necessary. On hypercomplex plane, the triality symmetry suggests that there are branching from $-\frac{1}{e}, -\frac{\omega}{e}, -\frac{\omega^2}{e}, (\omega = e^{\sqrt{-1}\pi/3})$ to infinity exist.

The superposition of pure quaternion allows the total spectra $1 + \omega + \omega^2 = 0$, but the in (3+1)D as in (2+1)D[16], chaos occurs. The spacelike time proposed by Tomonaga is important for analysis of paths which contain ordinary delay and TR delays.

Efficient algorithm for performing (3+1)D NDT whose medical application is anticipated is under investigation.

Acknowledgement

S.F. is grateful to the Laboratory for Industrial Research (Nissanken) for the financial aid to the travel expense to the ICU2025 Congress, Prof. M. Arai and Prof. K. Hamada for allowing the use of workstations as a research collaborator.

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